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Mathematical Methods for Economics

Michael Klein

Second Edition

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Part One

Introduction

This book begins with a three-chapter section that introduces some important concepts and tools that are used throughout the rest of the book. Chapter 1 presents background on the mathematical framework of economic analysis. In this chapter we discuss the advantages of using mathematical models in economics. We also introduce some characteristics of economic models. The discussion in this chapter makes reference to material presented in the rest of the book to put this discussion in context as well as to give you some idea of the types of topics addressed by this book.

Chapter 2 discusses the central topic of functions. The chapter begins by defining some terms and presenting some key concepts. Various properties of functions first introduced in this chapter appear again in later chapters. The final section of Chapter 2 presents a menu of different types of functions that are used frequently in economic analysis.

Two types of functions that are particularly important in economic analysis are exponential and logarithmic functions. As shown in Chapter 3, exponential functions are used for calculating growth and discounting. Logarithmic functions, which are related to exponential functions, have a number of properties that make them useful in economic modeling. Applications in this chapter, which include the distinction between annual and effective interest rates, calculating doubling time, and graphing time series of variables, demonstrate some of the uses of exponential and logarithmic functions in economic analysis. Later chapters make extensive use of these functions as well.

Chapter 1

The Mathematical Framework of Economic Analysis

What are the sources of long-run growth and prosperity in an economy? How does your level of education affect your lifetime earnings profile? Has foreign competition from developing countries widened the gap between the rich and the poor in industrialized countries? Will economic development lead to increased environmental degradation? How do college scholarship rules affect savings rates? What is the cost of inflation in an economy? What determines the price of foreign currency?

The answers to these and similar economic questions have important consequences. The importance of economic issues combined with the possibility for alternative modes of economic analysis result in widespread discussion and debate. This discussion and debate takes place in numerous forums including informal conversations, news shows, editorials in newspapers, and scholarly research articles addressed to an audience of trained economists. Participants in these discussions and debates base their analyses and arguments on implicit or explicit frameworks of reasoning.

Economists are trained in the use of explicit economic models to analyze economic issues. These models are usually expressed as sets of relationships that take a mathematical form. Thus an important part of an economist's training is acquiring a command of the mathematical tools and techniques used in constructing and solving economic models.

This book teaches the core set of these mathematical tools and techniques. The mathematics presented here provides access to a wide range of economic analysis and research. Yet a presentation of the mathematics alone is often insufficient for students who want to understand the use of these tools in economics because the link between mathematical theory and economic application is not always apparent. Therefore this book places the mathematical tools in the context of economic applications. These applications provide an important bridge between mathematical techniques and economic analysis and also demonstrate the range of uses of mathematics in economics.

The parallel presentation of mathematical techniques and economic applications serves several purposes. It reinforces the teaching of mathematics by providing a setting for using the techniques. Demonstrating the use of mathematics in economics helps develop mathematical comprehension as well as hone economic intuition. In this

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way, the study of mathematical methods used in economics as presented in this book complements your study in other economics courses. The economic applications in this book also help motivate the teaching of mathematics by emphasizing the practical use of mathematics in economic analysis. An effort is made to make the applications reference a wide range of topics by drawing from a cross section of disciplines within economics, including microeconomics, macroeconomics, economic growth, international trade, labor economics, environmental economics, and finance. In fact, each of the questions posed at the beginning of this chapter is the subject of an application in this book.

This chapter sets the stage for the rest of the book by discussing the nature of economic models and the role of mathematics in economic modeling. Section 1.1 discusses the link between a model and the phenomenon it attempts to explain. This section also discusses why economic analysis typically employs a mathematical framework. Section 1.2 discusses some characteristics of models used in economics and previews the material presented in the rest of the book.

1.1 ECONOMIC MODELS AND ECONOMIC REALITY

Any economic analysis is based upon some framework. This framework may be highly sophisticated, as with a multiequation model based on individuals who attempt to achieve an optimal outcome while facing a set of constraints, or it may be very simplistic and involve nothing more complicated than the notion that economic variables follow some well-defined pattern over time. An overall evaluation of an economic analysis requires an evaluation of the framework itself, a consideration of the accuracy and relevance of the facts and assumptions used in that framework, and a test of its predictions.

A framework based on a formal mathematical model has certain advantages. A mathematical model demands a logical rigor that may not be found in a less formal framework. Rigorous analysis need not be mathematical, but economic analysis lends itself to the use of mathematics because many of the underlying concepts in economics can be directly translated into a mathematical form. The concept of determining an economic equilibrium corresponds to the mathematical technique of solving systems of equations, the subject of Part Two of this book. Questions concerning how one variable responds to changes in the value of another variable, as embodied in economic concepts like price elasticity or marginal cost, can be given rigorous form through the use of differentiation, the subject of Part Three. Formal models that reflect the central concept of economics—the assumption that people strive to obtain the best possible outcome given certain constraints—can be solved using the mathematical techniques of constrained optimization. These are discussed in Part Four. Economic questions that involve consideration of the evolution of markets or economic conditions over time—questions that are important in such fields as macroeconomics, finance, and resource economics—can be addressed using the various types of mathematical techniques presented in Part Five.

While logical rigor ensures that conclusions follow from assumptions, it should also be the case that the conclusions of a model are not too sensitive to its assumptions.

It is typically the case that the assumptions of a formal mathematical model are explicit and transparent. Therefore a formal mathematical model often readily admits the sensitivity of its conclusions to its assumptions. The evolution of modern growth theory offers a good example of this.

A central question of economic growth concerns the long-run stability of market economies. In the wake of the Great Depression of the 1930s, Roy Harrod and Evsey Domar each developed models in which economies either were precariously balanced on a "knife-edge" of stable growth or were marked by ongoing instability. Robert Solow, in a paper published in the mid-1950s, showed how the instability of the Harrod–Domar model was a consequence of a single crucial assumption concerning production. Solow developed a model with a more realistic production relationship, which was characterized by a stable growth path. The Solow growth model has become one of the most influential and widely cited in economics. Applications in Chapters 8, 9, 13, and 15 in this text draw on Solow's important contribution. More recently, research on "endogenous growth" models has studied how alternative production relationships may lead to divergent economic performance across countries. Drawing on the endogenous growth literature, this book includes an application in Chapter 8 that discusses research by Robert Lucas on the proper specification of the production function as well as an application that presents a growth model with "poverty traps" in Chapter 13.[1]

Once a model is set up and its underlying assumptions specified, mathematical techniques often enable us to solve the model in a straightforward manner even if the underlying problem is complicated. Thus mathematics provides a set of powerful tools that enable economists to understand how complicated relationships are linked and exactly what conclusions follow from the assumptions and construction of the model. The solution to an economic model, in turn, may offer new or more subtle economic intuition. Many applications in this text illustrate this, including those on the incidence of a tax in Chapters 4 and 7, the allocation of time to different activities in Chapter 11, and prices in financial markets in Chapters 12 and 13. Optimal control theory, the subject of Chapter 15, provides another example of the power of mathematics to solve complicated questions. We discuss in Chapter 15 how optimal control theory, a mathematical technique developed in the 1950s, allowed economists to resolve long-standing questions concerning the price of capital.

A mathematical model often offers conclusions that are directly testable against data. These tests provide an empirical standard against which the model can be judged. The branch of economics concerned with using data to test economic hypotheses is called econometrics. While this book does not cover econometrics, a number of the applications show how to use mathematical tools to interpret econometric results. For example, in Chapter 7 we show how an appropriate mathematical function enables us to determine the link between national income per capita and infant mortality rates in

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[1] Solow's paper, "A contribution to the theory of economic growth," is published in the *Quarterly Journal of Economics,* 70, no. 1 (February 1956): 65–94. The other papers cited here are Roy F. Harrod, "An essay in dynamic theory," *Economic Journal,* 49 (June 1939): 14–33; Evsey Domar, "Capital expansion, rate of growth, and employment," *Econometrica,* 14 (April 1946): 137–147; and Robert Lucas, "Why doesn't capital flow from rich to poor countries?" *American Economic Review,* 80, no. 2 (May 1990): 92–96.

a cross section of countries. An application in Chapter 9 discusses some recent research on the relationship between pollution and income in a number of countries, which bears on the question of the extent to which rapidly growing countries will contribute to despoiling the environment. Chapter 8 includes an application that draws from a classic study of the financial returns to education.

It is natural to begin a book of this nature with a discussion of the many advantages of using a formal mathematical method for addressing economic issues. It is important, at the same time, to recognize possible drawbacks of this approach. Any mathematical model simplifies reality and, in so doing, may present an incomplete picture. The comparison of an economic model with a map is instructive here. A map necessarily simplifies the geography it attempts to describe. There is a trade-off between the comprehensiveness and readability of a map. The clutter of a very comprehensive map may make it difficult to read. The simplicity of a very readable map may come at the expense of omitting important landmarks, streets, or other geographic features. In much the same way, an economic model that is too comprehensive may not be tractable, while a model that is too simple may present a distorted view of reality.

The question then arises of which economic model should be used. To answer this question by continuing with our analogy to maps, we recognize that the best map for one purpose is probably not the best map for another purpose. A highly schematic subway map with a few lines may be the appropriate tool for navigating a city's subways, but it may be useless or even misleading if used aboveground. Likewise, a particular economic model may be appropriate for addressing some issues but not others. For example, the simple savings relationship posited in many economic growth models may be fine in that context but wholly inappropriate for more detailed studies of savings behavior.

The mathematical tools presented in this book will give you access to many interesting ideas in economics that are formalized through mathematical modeling. These tools are used in a wide range of economic models. While economic models may differ in many ways, they all share some common characteristics. We next turn to a discussion of these characteristics.

1.2 CHARACTERISTICS OF ECONOMIC MODELS

An economic model attempts to explain the behavior of a set of variables through the behavior of other variables and through the way the variables interact. The variables used in the model, which are themselves determined outside the context of the model, are called **exogenous variables.** The variables determined by the model are called **endogenous variables.** The economic model captures the link between the exogenous and endogenous variables.

A simple economic model illustrates the distinction between endogenous and exogenous variables. Consider a simple demand and supply analysis of the market for the familiar mythical good, the "widget." The endogenous variables in this model are the price of a widget and the quantity of widgets sold. The exogenous variables in this example include the price of the input to widget production and the price of the good that consumers consider as a possible substitute for widgets.

In this example there is an apparently straightforward separation of variables into the categories of exogenous and endogenous. This separation actually represents a central assumption of this model—the assumption that the market for the input used in producing widgets and the market for the potential substitute for widgets are not affected by what happens in the market for widgets. In general, the separation of variables into those that are exogenous and those that are endogenous reflects an important assumption of an economic model. Exogenous variables in some models may be endogenous variables in others. This may sometimes reflect the fact that one model is more complete than another in that it includes a wider set of endogenous variables. For example, investment is exogenous in the simplest Keynesian cross diagram and endogenous in the more complicated IS/LM model. In other cases the purpose of the model determines which variables are endogenous and which are exogenous. Government spending is usually considered exogenous in macroeconomic models but endogenous in public choice models. Even the weather, which is typically considered exogenous, may be endogenous in a model of the economic determinants of global warming. In fact, much debate in economics concerns whether certain variables are better characterized as exogenous or endogenous.

An economic model links its exogenous and endogenous variables through a set of relationships called **functions.** These functions may be described by specific equations or by more general relationships. Functions are defined in Chapter 2. In that chapter we describe different types of equations that are frequently used as functions in economic models. For now we identify three categories of relationships used in economic models: definitions, behavioral equations, and equilibrium conditions.

A **definition** is an expression in which one variable is defined to be identically equal to some function of one or more other variables. For example, profit (Π) is total revenue (TR) minus total cost (TC), and this definition can be written as

$$\Pi \equiv TR - TC,$$

where "$\equiv$" means "is identically equal to."

A **behavioral equation** represents a modeling of people's actions based on economic principles. The demand equation and supply equation in microeconomics, as well as the investment, money demand, and consumption equations in macroeconomics, all represent behavioral equations. Sometimes these equations reflect very basic economic assumptions such as utility maximization. In other cases, behavioral equations are not derived explicitly from basic economic assumptions but reflect a general relationship consistent with economic reasoning.

An **equilibrium condition** is a relationship that defines an **equilibrium** or **steady state** of the model. In equilibrium there are no economic forces within the context of the model that alter the values of the endogenous variables.

We use our example of the market for widgets to illustrate these concepts. The two behavioral equations in this model are a demand equation and a supply equation. We specify the demand equation for widgets as

$$Q^D = \alpha - \beta P + \gamma G$$